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Canonical vine copulas in the context of modern portfolio management: Are they worth it? ☆

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ABSTRACT

In the context of managing downside correlations, we examine the use of multi-dimensional elliptical and asymmetric copula models to forecast returns for portfolios with 3–12 constituents. Our analysis assumes that investors have no short-sales constraints and a utility function characterized by the minimization of Conditional Value-at-Risk (CVaR). We examine the efficient frontiers produced by each model and focus on comparing two methods for incorporating scalable asymmetric dependence structures across asset returns using the Archimedean Clayton copula in an out-of-sample, long-run multi-period setting. For portfolios of higher dimensions, we find that modeling asymmetries within the marginals and the dependence structure with the Clayton canonical vine copula (CVC) consistently produces the highest-ranked outcomes across a range of statistical and economic metrics when compared to other models incorporating elliptical or symmetric dependence structures. Accordingly, we conclude that CVC copulas are 'worth it' when managing larger portfolios.

1. Introduction

Equity returns suffer from increased correlations during bear markets (Longin and Solnik, 1995; Longin and Solnik, 2001; Ang and Chen, 2002). This characteristic, known as asymmetric or lower tail dependence, violates the assumption of elliptical dependence that is the basis of modern portfolio theory and mean–variance analysis (Ingersoll, 1987; Markowitz, 1952). While forecasting models incorporating asymmetric dependence produce significant gains for the investor with no short-sales constraints, they have been limited to bivariate or trivariate settings using standard Archimedean copulas (Patton, 2004; Garcia and Tsafack, 2011; Ba, 2011). More advanced flexible multivariate copulas ("vine copulas") introduced by Aas et al. (2009) presents an important opportunity for extending this literature further. Specifically, there are several interesting questions in the context of modern portfolio management. Does the more advanced Clayton canonical vine copula (CVC) produce economic and statistical outcomes superior to that of the Clayton standard copula (SC) in out-of-sample tests? Does the Clayton CVC exhibit superiority above some threshold size of portfolio? Does a more advanced model of the dependence structure produce outcomes superior to that of multivariate normality?

We answer these questions using an out-of-sample, long-run, multi-period investor horizon setting with portfolios comprising up to 12 US industry indices in a tactical asset allocation exercise. It is worth noting that our chosen focus on indices as "assets" delivers an important experimental advantage: collectively the full set of 12 indices constitutes the entire US market index. Thus, due to a binding dimensionality constraint, by employing indices as the basic constituents of the portfolios, our analysis is far more comprehensive than the alternative approach of using individual stocks. Moreover, as each index consists of hundreds of stocks, our investigation effectively involves highly diversified portfolios that exhibit low levels of idiosyncratic risk compared to other applications that form portfolios of individual stocks. Asymmetric dependence is evident regardless of whether an investor has a large number of US stocks within an equity investment portfolio (Ang and Chen, 2002) or is internationally diversified (Longin and Solnik, 2001; Longin and Solnik, 1995). Furthermore, Aggarwal and Aggarwal (1993) show that with 25 securities in a naive portfolio, the degree

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of negative skewness within the portfolio increases significantly and similar evidence is shown by Simkowitz and Beedles (1978) and Cromwell et al. (2000). Therefore, although diversification is prudent financial investment advice for 'normal' times, it becomes questionable when all stocks in the portfolio fall in times of market stress. Moreover, due to asymmetric dependence and negative skewness, the positive effects of diversification are greatly diminished when they are needed most (Chua et al., 2009). Thus, explicitly managing asymmetric dependence could be very worthwhile as investors might require additional compensation for undertaking downside risk (Ang et al., 2006), negative skewness (Simkowitz and Beedles, 1978; Cromwell et al., 2000) and have a preference for positively skewed portfolios (Arditti, 1967).

Our work is most relevant to Patton (2004) and Hatherley and Alcock (2007) who conduct studies upon portfolios of two and three assets respectively over investment horizons of less than a decade. Patton (2004) investigates whether asymmetries are predictable out-of-sample and portfolio decisions are improved by forecasting these asymmetries, as opposed to ignoring them over a single period investment horizon. He shows that investors with no short-sales constraints (i.e., portfolio weights are allowed to be negative) experience economic gains. Hatherley and Alcock (2007) report that managing asymmetric dependence, using a Clayton standard copula (SC) against the benchmark multivariate normal probability model, results in reduced downside exposure. Patton (2009) states that the obvious and perhaps most difficult avenue for future research is the extension of copula-based multivariate time series models to high dimensions. Such a breakthrough came with the CVC technology developed by Aas et al. (2009). The CVC consists of building blocks of pair copula and with a multitude of bivariate copulas from which to choose from, it is now possible to flexibly model the dependence structure for a multivariate joint distribution.

The novelty of our contribution lies in the non-trivial extension of this literature by incorporating methods that allow for higher scalability for capturing asymmetric dependence, with larger data sets over a multi-period investment horizon spanning several decades. Moreover, we apply a broad range of metrics to further investigate economic and statistical performance. We demonstrate how to meaningfully capture asymmetric dependence for higher portfolio dimensions by using the CVC model and mathematically expanding the SC.[1] A multi-period long-term investment horizon study is necessary as Barberis (2000) finds that multi-period decisions are substantially different from single-period decisions due to hedging demands if investment opportunities are time-varying (Merton, 1971). As investors might have different risk preferences, testing portfolio management strategies should include the application of a variety of risk-adjusted measures that incorporate downside risk and robustness against non-linear payoffs. Using an array of metrics to gauge portfolio performance is important as the presence of distributional asymmetries within asset returns can impact investors' portfolio choices (Harvey and Siddique, 2000; Longin and Solnik, 2001; Harvey et al., 2010). More specifically, our work manages asymmetric dependence by using the Clayton CVC that models asymmetric dependence of a portfolio of N assets with $N(N-1)/2$ parameters compared to the Clayton SC that employs just one parameter. Thus, the Clayton CVC, with its higher degree of parameterization, is capable of leading to superior forecasts of equity returns and improved portfolio management decisions. However, much of the forecasting literature indicates that more complicated models often provide poorer forecasts than simple and misspecified models (Swanson and White, 1997; Stock and Watson, 1999). Kritzman et al. (2010) state that practitioners often use simpler models to discriminate amongst investment opportunities, as complex econometric models can suffer from issues such as data mining, poor performance out-of-sample, and failure to produce meaningful profitability in a portfolio management context.

Given this background, our work leads to a deeper understanding of whether the increase in parameterization of an asset portfolio leads to both statistical and economically significant benefits. From a modeling viewpoint, the lower the dimensionality of a model, the higher the reliability of the parameters (Ané and Kharoubi, 2003). Furthermore, the main feature of the CVC compared to the SC is its mathematical scalability for portfolios of high dimensions. Thus, we seek insights into the portfolio size over which the model exhibits superiority. Furthermore, we assess whether the modeling of the dependence structure or the modeling of the marginals has the greater impact on a portfolio. This allows practitioners to understand the areas of a probability model that need to be analyzed further. We also demonstrate a method for building the CVC based on the sums of correlations of assets within the portfolio.

Our results show that for portfolios of 10 constituents and above, our most advanced model that captures asymmetries within the marginals and the dependence structure using the Clayton CVC consistently produces highly ranked outcomes across a range of statistical and economic performance metrics. Economic gains only exist for non-short sales constrained portfolios such as those used by hedge funds. In addition, it produces a returns distribution that exhibits significant positive skewness from a portfolio comprising industry indices that together represent the US market index. This is notable as US industry indices exhibit high levels of negative skewness. Our findings indicate that asymmetries should be incorporated in the modeling of both the marginals and the dependence structure and we find that modeling of asymmetries within the dependence structure has a greater impact than modeling of the marginals for portfolios of higher dimensions.

The paper is organized as follows. Section 2 describes the dataset. Section 3 details the methods used in modeling of the dependence structure and marginals, and the selection of the investor's utility function for portfolio optimization. Section 4 presents and discusses the empirical results of our study and we conclude in Section 5.

2. Data

Our data set consists of US monthly returns on 12 indices, constituting the full US market (data sourced from Ken French's website).[2] The indices are manufacturing (Manuf), other, money, chemicals (Chems), consumer non-durables (NoDur), retail (Shops), consumer durables (Durbl), business equipment, (BusEq), healthcare (Hlth), telecommunications (Telcm), utilities (Util), and energy (Enrgy). Similar to DeMiguel et al. (2009), we calculate arithmetic returns in excess of the US 1-month T-bill. The sample period extends from July 1963 to December 2010, yielding 570 observations in total. The first 120 observations are reserved for the parameterization process for our portfolio management strategy, while the out-of-sample period consists of 450 months from July 1973 to December 2010.

We implement our strategies in portfolios of three, six, nine, ten, eleven, and twelve constituents as shown in Table 1. All indices exhibit excess kurtosis and reject the null hypotheses for the Jarque–Bera test of normality at the 1% level. All indices exhibit negative skewness except for Durbl and Hlth. Durbl exhibits the minimum (−32.97%) and maximum (42.91%) return for our

[1] A detailed introduction to copula theory can be found in Joe (1997) and Nelsen (2006). Other resources for vine copula theory can be found in Aas et al. (2009) and Kurowicka and Joe (2011).

[2] The US market index is the value-weighted return on all NYSE, AMEX and NASDAQ stocks from CRSP. Industry indices are value-weighted returns formed by assigning each NYSE, AMEX, and NASDAQ stock from CRSP to an industry portfolio according to its 4-digit SIC code.

Table 1
Input data descriptive statistics.

Industry index	Mean	Std. deviation	Skewness	Kurtosis	Min	Max	Jarque–Bera
Manuf	0.55	0.054	−0.525	5.73	−29.15	21.55	203.12*
Other	0.39	0.056	−0.507	5.02	−29.92	18.80	121.71*
Money	0.51	0.055	−0.376	4.75	−22.40	20.51	87.43*
Chems	0.48	0.047	−0.240	5.16	−25.18	19.68	115.97*
NoDur	0.63	0.044	−0.296	5.08	−21.63	18.15	110.61*
Shops	0.57	0.053	−0.269	5.31	−28.91	25.22	133.19*
Durbl	0.42	0.063	0.146	8.31	−32.97	42.91	672.91*
BusEq	0.53	0.067	−0.213	4.14	−26.59	20.02	35.28*
Hlth	0.59	0.050	0.057	5.44	−21.07	29.07	142.20*
Telcm	0.39	0.047	−0.122	4.31	−15.97	21.98	42.10*
Utils	0.37	0.041	−0.077	4.04	−12.94	18.22	26.00*
Enrgy	0.66	0.054	−0.001	4.47	−19.10	23.33	50.88*

This table presents the descriptive statistics for excess monthly returns (relative to the 1-month T-bill) of 12 US industry indices (sourced from Ken French's website). The full sample runs from July 1963 to December 2010, yielding 570 observations. The indices are manufacturing (Manuf), other, money, chemicals (Chems), consumer non-durables (NoDur), retail (Shops), consumer durables (Durbl), business equipment, (BusEq), healthcare (Hlth), telecommunications (Telcm), utilities (Util), and energy (Enrgy).The mean, minimum and maximum are presented as percentages. Jarque–Bera tests the normality of the unconditional distribution of returns.
* Statistical significance at the 1% level.

Table 2
Unconditional sample correlations.

	Manuf	Other	Money	Chems	NoDur	Shops	Durbl	BusEq	Hlth	Telcm	Utils	Enrgy
Manuf	1.00											
Other	0.92	1.00										
Money	0.81	0.83	1.00									
Chems	0.87	0.82	0.77	1.00								
NoDur	0.79	0.79	0.81	0.82	1.00							
Shops	0.82	0.83	0.79	0.77	0.83	1.00						
Durbl	0.85	0.79	0.74	0.74	0.68	0.76	1.00					
BusEq	0.79	0.78	0.63	0.64	0.59	0.71	0.68	1.00				
Hlth	0.67	0.69	0.68	0.72	0.77	0.67	0.52	0.61	1.00			
Telcm	0.63	0.63	0.64	0.55	0.60	0.62	0.59	0.61	0.52	1.00		
Utils	0.53	0.53	0.60	0.53	0.61	0.46	0.45	0.31	0.47	0.50	1.00	
Enrgy	0.61	0.58	0.53	0.58	0.49	0.43	0.46	0.44	0.43	0.40	0.58	1.00

This table presents sample unconditional Pearson's correlations between monthly index returns for 12 US industries over the full sample period, July 1963 to December 2010. The indices are manufacturing (Manuf), other, money, chemicals (Chems), consumer non-durables (NoDur), retail (Shops), consumer durables (Durbl), business equipment, (BusEq), healthcare (Hlth), telecommunications (Telcm), utilities (Util), and energy (Enrgy).

sample. We report sample correlations in Table 2. The highest correlation occurs between Manuf and Other (0.92) while the lowest correlation occurs between Utils and BusEq (0.31).

The vine copula model by Aas et al. (2009) is a scalable methodology that can allow for large portfolios of assets but the user is constrained by the computational resources available. Our work involves the construction of investment portfolios consisting of indices as opposed to individual stocks. Modern portfolio theory suggests that such portfolios are more likely to exhibit elliptical dependence than are individual stocks. Thus, our analysis is biased against our empirical tests of portfolio optimization based on returns forecasts incorporating asymmetric dependence. Furthermore, by using indices we minimize other drawbacks that would occur with individual stocks- issues of size bias, selection bias, short-sales restrictions, idiosyncratic risk, higher transaction costs and illiquidity.

3. Research method

Portfolio management is a 2-stage process of (1) forecasting asset returns and (2) allocating weights to each asset within the portfolio (Markowitz, 1952). The portfolio weights are calculated based upon optimizing an investor's utility function for the portfolio asset returns forecasts. Intuitively, our research method follows the typical scenario faced by a portfolio manager in an investment fund. As new information arrives for each asset at month t, the portfolio manager has to make a forecast of asset returns for the next month $t+1$. Based upon the forecast of asset returns, the manager rebalances the weights to construct a portfolio that achieves the desired investment objective. The objective might be to achieve maximum utility based upon the investor's utility function or for the portfolio to maintain a fixed level of risk.

In the first stage of forecasting returns, similar to DeMiguel et al. (2009) we use a "rolling-window" approach. Each month t, starting from $t = W + 1$ uses the data within the previous W months (sample window = 120 months) to parameterize the multivariate probability distribution (detailed in subsection 3.1) and using Monte Carlo simulation[3] methods, 10,000 returns are produced using the Clayton Archimedean copula for each asset. These simulated data are used as a returns forecast. In the second stage, using these simulated data, we optimize a utility function defined by the minimization of Conditional Value-at-Risk (CVaR) (detailed in subsection 3.2) to calculate the "ideal" weights for the portfolio management strategy and apply it to each out-of-sample window month, $t+1$. Therefore, the calculated target weights are continually updated in each time period as they are dependent upon maximizing the investor's utility function based on the asset returns forecast.

3.1. Multivariate probability modeling

The cumulative distribution function (cdf) of a random vector can be expressed in terms of its component marginal distribution functions and a copula that describes the dependence structure

[3] We use the conditional sampling method for the SC as detailed in Cherubini et al. (2004, p.183). For the CVC, we use the Monte-Carlo simulation algorithm detailed by Aas et al. (2009).

between these components (Sklar, 1973). The copula approach is designed to use subjective judgement about marginal distributions, leaving all information relating to the dependence structure (as represented by the copula function) to be estimated separately. Thus, copulas allow the creation of multivariate distributions that have the flexibility required of risk management models and overcomes the limitations of the traditional multivariate models.

A copula function $C(u_1, u_2, \ldots, u_n)$ is defined as a cdf for a multivariate vector with support in $[0,1]^n$ and uniform marginals. The copula function is defined as:

$$C(u_1, u_2, \ldots, u_n) = P(U_1 \leqslant u_1, \ldots, U_n \leqslant u_n) \tag{1}$$

where $(U_1, U_2, \ldots, U_n)$ is the corresponding multivariate vector. Arbitrary marginal distribution functions may be selected such that by using the transformations $U_i = F_i(X_i)$, Eq. (2) defines a new multivariate distribution evaluated by $x_1, x_2, \ldots, x_n$ with marginals F_i, where $i = 1, 2, \ldots, n$. The copula function is defined in terms of cumulative distributive functions as shown:

$$F(x_1, x_2, \ldots, x_n) = C[F_1(x_1), F_2(x_2), \ldots, F_n(x_n)] \tag{2}$$

Sklar (1973) shows the converse of (2) where any multivariate distribution F can be written in terms of its marginals using a copula representation. It is possible to represent the density of the copula if we assume F_i and C to be differentiable. The joint density function $f(x_1, x_2, \ldots, x_n)$ is defined as:

$$\begin{aligned} f(x_1, x_2, \ldots, x_n) = & f_1(x_1) \times f_2(x_2) \times \cdots \times f_n(x_n) \\ & \times c[F_1(x_1), F_2(x_2), \ldots, F_n(x_n)], \end{aligned} \tag{3}$$

where the density of F_i is given by $f_i(x_i)$ and the density of the copula is given by:

$$c(u_1, u_2, \ldots, u_n) = \frac{\partial^n C(u_1, u_2, \ldots, u_n)}{\partial u_1 \partial u_2 \ldots \partial u_n} \tag{4}$$

As can be seen from Eq. (3), under appropriate conditions, the joint density can be written as a product of the marginal densities and the copula density, as opposed to the traditional modeling approaches where the joint density is decomposed into a product of marginal and conditional densities. The dependence structure among the X_i's is captured by the density $C(u_1, u_2, \ldots, u_n)$ while the f_i's capture the behavior of the marginals. The copula is chosen to select the dependence between asset returns and is able to account for asymmetric and symmetric correlation structures depending upon the copula chosen.

3.1.1. Clayton Archimedean copula

Archimedean copulas are commonly used due to their flexibility and usefulness in modeling complex dependence structures from a generator function as shown in Eq. (5):

$$C(u_1, u_2, \ldots, u_n) = G^{-1}[G(u_1) + G(u_2) + \cdots + G(u_n)] \tag{5}$$

We use the Clayton copula due to its ability to parameterize lower tail dependence across asset returns. Fig. 1 is a plot of US monthly market returns against 12 counterpart US industry index returns from July 1963 to December 2010 (in excess of the 1-month T-bill rate). The 'fan-shape' behavior exhibited by the index and market returns is indicative of an asymmetric (lower tail) dependence structure. The boxed regions in Fig. 1 show that the correlations between the aggregate US market and its constituent indices are higher (lower) when returns are below (above) the market monthly threshold return of −20% (+20%), as the density of points on the downside is greater than that on the upside (indeed the upside box is totally vacant). This pattern corroborates other studies (e.g., Longin and Solnik, 1995; Ang and Chen, 2002; Patton, 2004) that describe these conditional correlations as "exceedance correlations". Our application of the Clayton copula

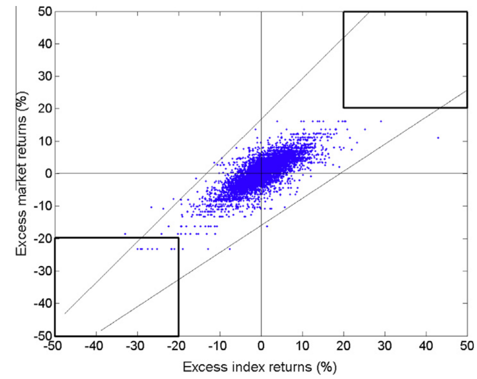


Fig. 1. Empirical relation between the US market and industry indices This figure plots monthly excess returns for the US market vs. 12 constituent industry indices from July 1963 to December 2010 (in excess of the US 1-month T-bill). The boxed regions highlight threshold return values above +20% and below −20% for the industry indices and the US market.

is such that lower tail dependence across equity returns is accommodated but not imposed. By focusing on managing the scenario of lower tail dependence, we seek to design a portfolio management strategy capable of providing reliably good performance during times of market stress (Chua et al., 2009).

Fig. 2 shows a variety of dependence structures for the bivariate case. If the Clayton Copula or the Pearson's correlation parameters are close to zero, the dependence structure is circular as shown in Fig. 2a. Elliptical (Fig. 2b) and asymmetric (Fig. 2c) dependence structures are accommodated but not imposed by the covariance matrix and Clayton copula, respectively. It can be seen that the Clayton copula, with its ability to parameterize lower tail dependence is a more appealing model of actual returns in the long run when compared to the assumptions of Mean–Variance Portfolio Theory (MVPT) where asset returns are assumed to have elliptical dependence as shown by Fig. 2b.

Substituting the Clayton copula generator function into Eq. (5) and using Eq. (4), for illustrative purposes, we can generate a Clayton SC probability distribution function (pdf) for six assets as shown:

$$\begin{aligned} c_{123456} &= C_\alpha(u_1, u_2, \ldots, u_6) \\ &= \left(u_1^{-\alpha} + u_2^{-\alpha} + u_3^{-\alpha} + u_4^{-\alpha} + u_5^{-\alpha} + u_6^{-\alpha} - 5\right)^{\frac{1}{\alpha}+6} \end{aligned} \tag{6}$$

Thus, lower tail dependence across six assets is characterized by a single parameter α. This results in a multivariate probability distribution of the form given by Eq. (7) where f_n denotes the marginal pdf and c_n is the copula pdf:

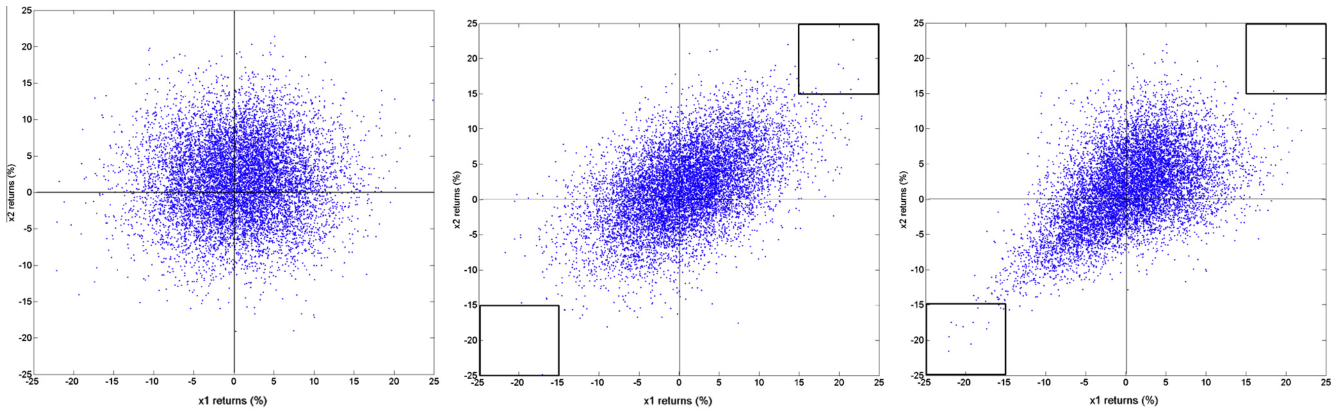
$$f_{12345} = f_1 \cdot f_2 \cdot f_3 \cdot f_4 \cdot f_5 \cdot f_6 \cdot c_{123456} \tag{7}$$

3.1.2. Canonical vine copula

Conventionally, a copula model is limited to a 1-parameter or 2-parameter specification of the dependence structure, which represents a potentially severe empirical constraint. Clearly, when modeling the joint distribution of multiple assets, such limited parameter models are unlikely to adequately capture the dependence structure. Moreover, the Gaussian copula lacks tail dependence and even though the multivariate Student t copula is able to generate different tail dependence[4] for each pair of variables, it imposes the same upper and lower tail dependence across all pairs. These limitations are overcome by the canonical vine model by building bivariate copulas of conditional distributions.

Canonical vine copulas are flexible multivariate copulas that are generated via hierarchical construction and can be decomposed

[4] Tail dependence for a multivariate Student t copula is a function of the correlation and degrees of freedom.



(a) $\alpha = 0.01$ or $\rho = 0.01$ (b) Pearson's correlation, $\rho = 0.5$ (c) Clayton Copula, $\alpha = 0.9$

Fig. 2. Plots illustrating alternative dependence structures This figure shows a variety of dependence structures for the bivariate case of x_1 and x_2 asset returns. Fig. 2a shows the circular dependence structure when the Pearson's correlation and Clayton copula parameters are close to zero. Fig. 2b shows the elliptical dependence structure produced for a correlation parameter of 0.5. Fig. 2c shows the asymmetric (lower tail) dependence structure produced for an Clayton copula parameter of 0.9.

into a cascade of bivariate copulas. The principle is to model dependence using simple local building blocks (pair-copulas) based on conditional independence. A joint probability density function of n variables $u_1, u_2, \ldots, u_n$ can be decomposed without loss of generality by iteratively conditioning as shown in Eq. (8):

$$f(u_1, u_2, \ldots, u_n) = f(u_1) \cdot f(u_2|u_1) \cdot f(u_3|u_1, u_2) \cdots f(u_n|u_1, \ldots, u_{n-1}) \tag{8}$$

Each of the terms in this product can be decomposed further using conditional copulas. For example, the first conditional density can be decomposed into the copula function c_{12} (the copula linking u_1 and u_2) multiplied by the density of u_2 as shown in (9) where $F_i(.)$ is the cdf of u_i:

$$f(u_2|u_1) = c_{12}[F_1(u_1), F_2(u_2)]f_2(u_2) \tag{9}$$

Thus, the joint density of the three-dimensional case can be represented by a function of the bivariate conditional copulas and the marginal densities:

$$\begin{aligned} f(u_1, u_2, u_3) = {} & c_{23|1}(F_{2|1}(u_2|u_1), F_{3|1}(u_3|u_1))c_{12}(F_1(u_1), F_2(u_2)) \\ & c_{13}(F_1(u_1), F_3(u_3))f_1(u_1)f_2(u_2)f_3(u_3) \end{aligned} \tag{10}$$

Joe (1997) proves that conditional distribution functions may be solved using (11):

$$F(u|v) = \frac{\partial C_{u,v_j|v_{-j}}(F(u|v_{-j}, F(v_j|v_{-j})))}{\partial F(v_j|v_{-j})} \tag{11}$$

where v_{-j} is the vector v that excludes the component v_j.

While other vine specifications exist, such as the D-vine case (Aas et al., 2009), we select the canonical vine alternative due to the efficiency of its hierarchical structure. If key variables that govern the interactions in the data set can be identified during the modeling process, it is possible to locate these variables towards the root of the canonical vine. Thus, we are able to build the canonical vine by ordering assets closer to the root of the structure by their degree of correlation with other assets within the portfolio.

If the Clayton CVC is implemented, the dependence structure of a portfolio of six assets would be parameterized with 15 pairwise copula parameters.[5] As a result, the multivariate probability distribution for the six asset case is as shown in Eq. (12) where f_n denotes the marginals pdf and c_n denotes the pairwise copula pdfs:

$$\begin{aligned} f_{123456} = {} & f_1 \cdot f_2 \cdot f_3 \cdot f_4 \cdot f_5 \cdot f_6 \cdot c_{12} \cdot c_{13} \cdot c_{14} \cdot c_{15} \cdot c_{16} \cdot c_{23|1} \\ & \cdot c_{24|1} \cdot c_{25|1} \cdot c_{26|1} \cdot c_{34|12} \cdot c_{35|12} \cdot c_{36|12} \cdot c_{45|123} \\ & \cdot c_{46|123} \cdot c_{56|1234} \end{aligned} \tag{12}$$

3.1.3. Marginals modeling

We model the marginals using two alternative methods. First, to establish a baseline, we assume that they adhere to a normal distribution. However, assumptions of normality within the marginals can lead to the copula model capturing asymmetries within the marginals. Thus, for further comparison between the standard copula and the vine copula model, the marginal distributions are also modeled using the univariate skewed Student t (Skew-T) setup of Hansen (1994) with constant unconditional mean and variance:[6]

$$y_{i,t} = c_{i,t} + \sqrt{h_{i,t}} \cdot z_{i,t}, \quad \text{for } i = 1, \ldots, n, \tag{13}$$

$$h_{i,t} = \omega_i, \tag{14}$$

$$z_{i,t} \sim \text{ skewed Student } t(v_i, \lambda_i), \tag{15}$$

where the skewed Student t density is given by

$$g(z|v,\lambda) = \begin{cases} bc\left(1 + \frac{1}{v-2}\left(\frac{bz+a}{1-\lambda}\right)^2\right)^{-(v+1)/2} & z < -a/b, \\ bc\left(1 + \frac{1}{v-2}\left(\frac{bz+a}{1+\lambda}\right)^2\right)^{-(v+1)/2} & z \geqslant -a/b. \end{cases} \tag{16}$$

The constants a,b, and c are defined as

$$a = 4\lambda c\left(\frac{v-2}{v-1}\right), b^2 = 1 + 3\lambda^2 - a^2, c = \frac{\Gamma\left(\frac{v+1}{2}\right)}{\sqrt{\pi(v-2)}\Gamma\left(\frac{v}{2}\right)} \tag{17}$$

Using a skewed marginal distribution gives greater confidence that any asymmetry found in the dependence structure truly reflects dependence and cannot be attributed to poor modeling of

[5] The number of parameters required to parameterize a dependence structure using a canonical vine model given a k-parameter copula is given by $k\sum_{i=1}^{N-1} i$ where N is the number of assets.

[6] Although return variance is known to be heteroscedastic, our study makes the simplifying assumption of homoscedasticity. This allows us to focus upon performance improvements by incorporating asymmetries into the dependence structure and marginal distribution, as these are identified to have a greater impact on the portfolio selection process (Scott and Horvath, 1980; Kane, 1982; Peiro, 1999; Harvey et al., 2010). In untabulated analysis, when we incorporate asymmetric volatility in the marginals, using the GARCH-GJR model (Glosten et al., 1993), we find no qualitative difference in results. Details (available from the authors upon request) are suppressed to conserve space.

the marginals. During bear markets we are likely to observe a higher incidence of large negative returns than of large positive returns (of similar magnitude) in a booming market. Thus, we expect this observation to be captured by a negative λ (indicating a left-skewed density).

3.1.4. Portfolio parameterization process

Given that the Clayton SC parameterizes lower tail dependence with a single parameter, the order of the assets entering the portfolio has no impact on the modeling process. In contrast, the hierarchical structure of the canonical vine copula means that the ordering is important. Accordingly, we design the canonical vine structure by placing assets that have the highest degree of linear correlation with all the other assets in the sample window at the 'root' of the structure. This is achieved by calculating and summing the Pearson's correlations between all assets during the sample window. More formally, we define the correlation metric in Eq. (18):

$$\Theta_y = \sum_{x=1}^{N} \theta_{xy}, \text{ where } x, y \in N \tag{18}$$

θ_{xy} is an $N \times N$ matrix of the Pearson's correlation parameter of the monthly returns between each pair of assets x and y that are both part of our portfolio of N assets. Θ_y is a $N \times 1$ matrix where each element is the sum of the Pearson's correlation parameter of y with all other assets x. The largest value in Θ_y has the highest absolute linear correlation with all other assets within the portfolio during the sample window and is placed at the root of the hierarchical structure of the canonical vine.

In our application of the model, we use maximum likelihood estimation. Due to the large number of parameters that need to be incorporated in our model, numerical maximization of the likelihood function is difficult and requires substantial computer resources. For example, for each of the 450 out-of-sample time periods, a portfolio of 12 assets modeled using the Clayton CVC with univariate skewed Student t marginals requires an estimate of 114 parameters.[7] We use the inference for margins (IFM) (Joe, 1997) method that is a 2-step parametric estimation procedure where the copula and marginal distribution parameters are estimated separately.

3.2. Optimization of the investor's utility function

Given that our focus is on lower tail dependence, it makes sense to select an optimization strategy that has a meaningful downside risk emphasis. Accordingly, we choose to minimize CVaR in preference to Value-at-Risk (VaR), as the former metric is considered to be a coherent risk measure (Uryasev, 2000). It is suitable for an investor who is focused on minimizing downside risk and is indifferent (or might even prefer) upside variance. Furthermore, it generates an efficient frontier that incorporates non-normality. Thus, if asset returns exhibit lower tail dependence, more emphasis will be placed on reducing this risk in comparison to MVPT portfolios that assume quadratic utility and ignore all higher moments of the returns distribution. Optimizing portfolios to reduce CVaR is a linear programming exercise and leads to lower values of VaR. Rockafellar and Uryasev (2000) present CVaR in integral form as shown below:

$$\phi_\beta(\mathbf{w}) = \frac{1}{1-\beta} \int_{f(\mathbf{w},\mathbf{r}) \geqslant \alpha_\beta(\mathbf{w})} f(\mathbf{w},\mathbf{r}) p(\mathbf{r}) d\mathbf{r} \tag{19}$$

where a loss function is presented by $f(\mathbf{w}, \mathbf{r})$ and the probability that $\mathbf{r}$ occurs is p ($\mathbf{r}$). In addition, they show that, when Monte Carlo integration is used, Eq. (20) is a suitable approximation to minimize CVaR for a given level of return:

$$\min_{(\mathbf{w},\alpha)} \quad F_\alpha(\mathbf{w},\beta) = \alpha + \frac{1}{q(1-\beta)} \sum_{k=1}^{q} [-\mathbf{w}^T \mathbf{r}_k - \alpha]^+ \tag{20}$$

where

$$\mu(\mathbf{w}) \leqslant -R, \tag{21}$$

$$w^T \mathbf{1} = 1, \tag{22}$$

q represents the number of samples generated by Monte Carlo simulation, α represents VaR and $\mathbf{1}$ is a vector of ones. β represents the threshold value usually set at 0.99 or 0.95 and $\mathbf{r}_k$ is the k^{th} vector of simulated returns. The vector of portfolio weights, $\mathbf{w}$, is extracted from the optimization procedure to generate the portfolio that minimizes CVaR for a given R. As we consider the investor who is averse to extreme downside losses, we set β to 0.99-analogous to an investor who wishes to minimize losses at the 1% level of CVaR, similar to Basel (2004) requirements.

4. Results

We investigate the applicability of the different multivariate probability models in the context of investors who wish to minimize the event of extreme losses within their portfolio. First, we perform an in-sample study to observe the efficient frontiers produced from a range of probability models and also from historical data of index excess returns for portfolios of different sizes. We perform this analysis to observe which probability model produces an efficient frontier closest to that of historical returns. Second, we perform a multi-period, long-term, out-of-sample study which uses the probability models as returns forecasts and optimize our portfolios to minimize CVaR. We use a wide range of statistical and economic metrics, including VaR backtests, to assess the superiority of each model in an out-of-sample portfolio management context.

4.1. Efficient frontiers, E (R) vs. CVaR

Fig. 3 shows the efficient frontiers produced when we apply simulated returns data generated from several multivariate probability models, and historical excess returns data, over the entire sample period from June 1963 to December 2010 without constraining short-sales. Annual expected returns are plotted against CVaR as a measure of risk. We use an in-sample analysis to investigate which multivariate probability model produces an efficient frontier closest to that of the historical data of excess index returns.

The copulas used are (1) Clayton SC; (2) Clayton CVC; (3) Student t; and (4) Gaussian. The marginals are either modeled as a normal (Normal) or with unconditional mean and variance and incorporating skewness and kurtosis using the model of Hansen (1994) within the resulting error distribution (Skew-T). The multivariate normal (MVN) model is included as a benchmark to indicate the resulting efficient frontier based on the assumptions of MVPT where the marginals are normal and the elliptical dependence structure is defined by the covariance matrix.

For the 3-asset case, we find clustering of efficient frontiers largely based on whether the marginals are normal or Skew-T. Models with normal marginals under-estimate risk relative to the Skew-T. Furthermore, elliptical dependence models with normal marginals generate efficient frontiers that cluster together (e.g., MVN, Student t Normal, Gaussian Normal) and tend to under-estimate risk relative to the models incorporating asymmetries

[7] For the 12 asset case, a Clayton CVC requires an estimate of 66 pair wise correlation parameters. Each of the Skew-T marginal distributions require an estimate of 4 parameters namely, the unconditional mean, unconditional variance, skewness, and degrees of freedom.

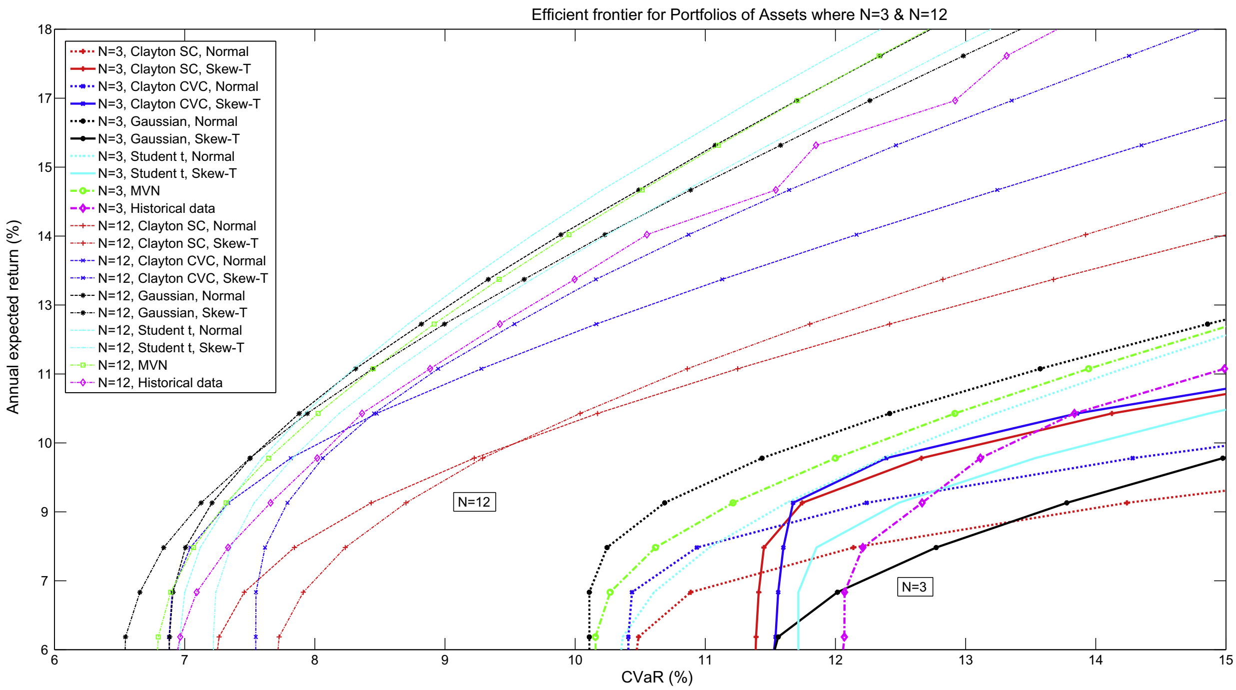


Fig. 3. Alternative efficient frontiers (annual E (R) vs. CVaR) This figure shows the annual expected returns against CVaR for portfolios of 3 and 12 assets. Nine alternative strategies are implemented for each portfolio using the full sample from July 1963 to December 2010. Clayton SC and Clayton CVC models allow for asymmetric dependence. Models that allow for elliptical dependence structures are the Gaussian, Student *t* copulas, and MVN models. The Skew-T model allows for asymmetries within the marginals while the Normal model does not.

either in the dependence structure or in the marginals. However, when elliptical dependence structures are modeled with Skew-T marginals (e.g., Student *t* Skew-T, Gaussian Skew-T), the efficient frontier changes considerably. These findings are consistent with the intuition that in the case of modeling small portfolios, modeling of the marginals will have a greater effect on the portfolio than the dependence structure. When Skew-T marginals are used, only asymmetries within the dependence structure need to be modeled by the copula and in the case of a small portfolio, there are no significant improvements to be gained by using the CVC. The efficient frontier produced by using historical returns is closer to the models that incorporate asymmetries within the marginals with the Skew-T.

For the 12-asset case, models incorporating elliptical dependence structures cluster together (e.g., MVN, Gaussian, Student *t*) even though Skew-T marginals are used. Visibly different efficient frontiers are present for the Clayton CVC and SC even when Skew-T or normal marginals are used. Elliptical dependence structures generally underestimate the level of risk compared to asymmetric dependence structures. For elliptical dependence models, normal marginals underestimate the level of risk relative to Skew-T marginals. In an opposing fashion, for asymmetric dependence models, normal marginals produce efficient frontiers that overestimate the level of risk relative to Skew-T marginals. As the mathematical model that generates an efficient frontier closest to that produced by historical returns data is the most suitable in a portfolio and risk management context, this is indicative that the Clayton CVC model with Skew-T marginals provides the closest fit in the 12-asset case.

Similar to modern portfolio theory for large portfolios, modeling of asymmetry within the dependence structure has a much greater impact on the efficient frontier than just incorporating it in the modeling of the marginals. The diversification effect is visible as the efficient frontier for 12 assets dominates each 3-asset counterpart, irrespective of the model being used. The efficient frontiers produced by historical returns for portfolios of 12 and 3 assets are closest to models that allow for asymmetries in the dependence structure and marginals. Therefore, from an in-sample perspective, multivariate probability models that do not incorporate these asymmetries can produce less reliable forecasts.

4.2. Out-of-sample portfolio performance

We study the use of multivariate probability models incorporating asymmetries within the dependence structure or marginals in portfolio asset returns forecasts. We then optimize our portfolios to minimize CVaR. This out-of-sample analysis is performed in a long-run, multi-period investor horizon from June 1973 to December 2010. We investigate the Clayton SC with normal marginals (SC-N), Clayton CVC with normal marginals (CVC-N), Clayton SC with Skew-T marginals (SC-S), and Clayton CVC with Skew-T marginals (CVC-S). We explore their performance out-of-sample in relation to each other and against the benchmark of the MVN probability model. As the SC-N, CVC-N, SC-S, and CVC-S models have in common the incorporation of returns asymmetry in some form, we will refer to them collectively as asymmetric returns (AR) models. As before, all portfolio strategies are not short-sales constrained.

4.2.1. Descriptive statistics of portfolio strategies

Table 3 shows the descriptive statistics of the returns distribution for each of the five portfolio strategies-we report mean, standard deviation, skewness, kurtosis, minimum value, maximum value and maximum drawdown. Maximum drawdown is the largest peak-to-trough percentage drop in returns during the investment period. As the size of the portfolio increases, across all portfolio strategies, the mean tends to improve (the one contrary case is SC-N) and the standard deviation decreases. This supports

Table 3
Out-of-sample copula-based portfolio strategy descriptive statistics.

Metric	Method	Portfolio size, N					
		3	6	9	10	11	12
Mean	SC-N	0.57	−0.01	0.20	0.14	0.04	0.29
	CVC-N	0.35	0.23	0.49	0.36	0.26	0.42
	SC-S	0.33	0.06	0.09	0.32	0.10	0.48
	CVC-S	0.22	0.26	0.35	0.41	0.41	0.52
	MVN	0.53	0.20	0.38	0.27	0.23	0.42
Std. dev.	SC-N	0.090	0.079	0.068	0.067	0.063	0.061
	CVC-N	0.085	0.066	0.068	0.063	0.059	0.055
	SC-S	0.088	0.074	0.068	0.067	0.062	0.061
	CVC-S	0.083	0.068	0.065	0.061	0.060	0.055
	MVN	0.086	0.057	0.058	0.054	0.053	0.050
Skewness	SC-N	−0.30	1.55	0.86	0.93	0.86	1.42
	CVC-N	−0.49	−0.67	1.08	0.67	0.83	1.02
	SC-S	−0.31	−0.74	0.73	1.05	0.73	1.09
	CVC-S	−0.59	−0.75	0.76	1.11	1.52	1.20
	MVN	−0.34	−0.43	0.61	0.38	0.66	0.30
Kurtosis	SC-N	8.62	27.34	11.77	12.67	13.63	17.51
	CVC-N	7.17	5.55	13.48	12.13	17.12	15.32
	SC-S	6.67	8.14	13.37	14.68	10.91	14.41
	CVC-S	5.96	6.80	13.69	17.46	23.59	17.85
	MVN	8.62	5.05	9.95	8.32	11.00	8.57
Min	SC-N	−47.55	−33.79	−22.29	−25.42	−23.78	−22.00
	CVC-N	−47.55	−26.54	−23.60	−23.22	−29.26	−21.03
	SC-S	−41.73	−37.74	−26.23	−27.46	−23.62	−24.49
	CVC-S	−41.53	−31.14	−28.79	−23.45	−22.71	−20.83
	MVN	−47.55	−22.41	−20.80	−18.48	−17.72	−17.74
Max	SC-N	50.78	80.26	52.22	51.80	49.92	54.41
	CVC-N	35.34	25.08	54.28	48.76	51.32	46.81
	SC-S	41.99	34.08	55.32	55.00	45.93	52.27
	CVC-S	31.17	31.46	53.94	54.74	59.33	50.05
	MVN	42.68	21.84	41.64	36.77	41.05	35.23
Max. drawdown	SC-N	94.15	98.30	94.37	94.24	93.47	77.78
	CVC-N	91.15	96.61	88.49	91.25	90.65	75.05
	SC-S	87.95	96.99	93.88	87.13	91.58	68.01
	CVC-S	96.10	94.54	88.91	81.40	85.63	65.65
	MVN	91.80	93.72	86.09	87.05	87.68	70.49

This table reports a statistical overview of the returns distributions generated by each portfolio strategy out-of-sample. The mean, minimum, maximum, and maximum drawdown are presented as percentages. SC-N is the Clayton standard copula (SC) with normal marginals, CVC-N is the Clayton canonical vine copula (CVC) with normal marginals, SC-S is the Clayton SC with Skew-T marginals, CVC-S is the Clayton CVC with Skew-T marginals and MVN is the multivariate normal model (benchmark case).

the maxim that increasing the number of assets in an investment portfolio achieves diversification benefits.

SC-N produces the highest mean when there are three assets. The most advanced model, CVC-S, produces the highest mean for 6, 10, 11, and 12 assets, while CVC-N produces the highest mean for the nine assets case. These results are indicative that models that account for asymmetry in some way, outperform the benchmark model (MVN), in terms of a higher mean value. Notably, the mean for SC-N under performs the MVN benchmark beyond three assets, suggesting that the covariance matrix is able to capture the dependence model in a superior fashion. As the number of assets within the portfolio increases, the SC parameter asymptotes towards zero and is unable to capture asymmetric dependence sufficiently. The CVC models (CVC-N and CVC-S) are not restricted in this fashion mathematically and produce higher means than SC-N for portfolios of six assets and above. Similarly, the SC-S produces higher means than the SC-N as the former captures asymmetries within the marginals whereas the SC-N does not.

MVN exhibits the lowest standard deviation for portfolios comprising six assets and above. SC models exhibit higher standard deviation than the CVC models for all portfolio sizes. However, the higher standard deviations exhibited by the AR models could be indicative of a larger upside variance that is desirable for loss-averse investors. Therefore, we explore the performance of the strategies in relation to a range of downside risk measures in Section 4.2.2. For portfolios exceeding nine assets, CVC-S consistently produces highly positively skewed returns-a desirable attribute, especially given that (a) the constituent indices in our dataset largely exhibit negative skewness and (b) investors are likely to prefer positively-skewed portfolios. In addition, CVC-S also exhibits the lowest maximum drawdown for portfolios of 10 assets and above. For any given portfolio size, a strategy that incorporates asymmetry in some form exhibits returns with a much higher level of skewness and kurtosis compared to the benchmark model (MVN). Analysis of the minimum and maximum returns show that the high levels of kurtosis are largely attributed to the exposure of the SC and CVC to large maximum returns. Finally, we observe that CVC-S produces the highest mean, skewness (second highest for portfolio size 12) and lowest maximum drawdown for portfolios of 10 assets and above.

4.2.2. Risk-adjusted performance

Table 4 reports a range of risk-adjusted measures used to assess the out-of-sample performance of each portfolio management strategy. The Sharpe ratio penalizes the entire standard deviation of portfolio returns, whereas the Sortino ratio penalizes only downside standard deviation. The Omega ratio is a practical measure that makes no assumptions regarding investor risk preferences or utility functions except that investors prefer more to

Table 4
Out-of-sample risk-adjusted performance of copula-based portfolio strategies.

Risk-adjusted metric	Method	Portfolio size, N					
		3	6	9	10	11	12
Sharpe ratio	SC-N	0.064	−0.001	0.030	0.020	0.007	0.048
	CVC-N	0.041	0.035	0.073	0.057	0.045	0.076
	SC-S	0.037	0.008	0.014	0.048	0.016	0.079
	CVC-S	0.027	0.038	0.054	0.067	0.068	0.094
	MVN	0.061	0.035	0.066	0.050	0.043	0.084
Sortino ratio	SC-N	0.091	−0.002	0.044	0.030	0.009	0.075
	CVC-N	0.057	0.046	0.113	0.085	0.066	0.117
	SC-S	0.052	0.010	0.019	0.073	0.023	0.123
	CVC-S	0.037	0.051	0.080	0.101	0.103	0.146
	MVN	0.088	0.048	0.100	0.074	0.064	0.124
Omega ratio	SC-N	1.201	0.996	1.090	1.061	1.019	1.148
	CVC-N	1.122	1.100	1.238	1.181	1.142	1.248
	SC-S	1.109	1.023	1.040	1.154	1.046	1.248
	CVC-S	1.076	1.115	1.166	1.211	1.223	1.306
	MVN	1.194	1.099	1.202	1.148	1.127	1.254
MPPM	SC-N	−0.037	−0.075	−0.030	−0.037	−0.042	−0.007
	CVC-N	−0.054	−0.028	0.006	−0.003	−0.010	0.015
	SC-S	−0.060	−0.066	−0.044	−0.014	−0.034	0.014
	CVC-S	−0.064	−0.028	−0.008	0.006	0.007	0.027
	MVN	−0.034	−0.016	0.006	−0.002	−0.006	0.020
Mean/VaR 1%	SC-N	2.31	−0.04	1.04	0.73	0.21	2.21
	CVC-N	1.41	0.97	2.61	1.87	1.50	2.71
	SC-S	1.29	0.21	0.46	1.74	0.62	3.24
	CVC-S	0.86	1.05	1.83	2.54	2.25	3.81
	MVN	2.25	1.20	2.57	2.02	1.74	3.23
Mean/CVaR 1%	SC-N	1.63	−0.04	0.97	0.62	0.19	1.63
	CVC-N	1.07	0.92	2.45	1.72	1.24	2.32
	SC-S	0.99	0.18	0.40	1.48	0.51	2.70
	CVC-S	0.70	0.97	1.64	2.15	1.99	3.04
	MVN	1.52	1.00	2.13	1.66	1.43	2.68

This table reports a range of risk-adjusted measures for the portfolio management strategies out-of-sample. The Sharpe (Sortino) ratio is the ratio of mean excess return to the total standard deviation (lower partial moment) of the portfolio. The Omega ratio is a probability weighted ratio of gains to losses relative to a threshold value of 0%. The Manipulation Proof Performance Measure (MPPM) is a metric that is robust against portfolios with non-linear payoffs. Mean/VaR and Mean/CVaR capture portfolio performance relative to extreme downside risk at the 1% level. SC-N is the Clayton standard copula (SC) with normal marginals, CVC-N is the Clayton canonical vine copula (CVC) with normal marginals, SC-S is the Clayton SC with Skew-T marginals, CVC-S is the Clayton CVC with Skew-T marginals and MVN is the multivariate normal model (benchmark case).

less. The MPPM is a portfolio ranking metric developed by Goetzmann et al. (2007) for robustness against non-linear payoffs from managed portfolios.[8] The Mean/CVaR and Mean/VaR metrics measure portfolio performance relative to extreme downside risk exposure at the 1% level. The best performing strategies exhibit the highest values for each measure.

Across all portfolio sizes, at least one of the AR models outperform the MVN benchmark for the Sharpe, Sortino and Omega metrics. For the MPPM metric, this result holds true for portfolios of nine assets and above. This is also generally the case for Mean/VaR and Mean/CVaR except for a portfolio of six assets where MVN produces the highest value. Thus, based on our analysis models incorporating return asymmetries in some form, outperform the MVN benchmark.

Across the Sharpe, Sortino and Omega Ratios, it is observed that as the portfolio increases in size, so does the level of complexity required in the model to produce the highest ranked outcome. For portfolio sizes above three assets, the CVC model outperforms its counterpart SC model across the Sharpe, Sortino and Omega ratios. While SC-N excels in our smallest portfolio setting, it performs very poorly across all metrics for portfolios of six assets and above. This finding suggests that the Clayton SC is too simplistic and is unable to meaningfully capture asymmetric dependence CVC-S excels in large portfolios of 10 assets and above and performs poorly for small portfolios of three assets. This contrast is potentially due to the fact that there is little or no benefit to be gained by using a complex model of the dependence structure and marginals for simpler, smaller portfolios. In such cases, using advanced models induces estimation error which swamps any benefits from the modeling, resulting in poor portfolio decisions.

For the MPPM, Mean/VaR and Mean/CVaR metrics, CVC-S consistently produces the highest ranked outcomes for portfolios of 10 assets and above. The latter two are indicative that this CVC method is able to produce a higher return without a substantial increase in downside exposure. At less than 10 assets, MVN produces the highest ranked MPPM values but some of these values are negative due to non-linear payoffs.

Generally, these results support the view that increases in model complexity and parameterization for small portfolios have little or even negative benefits due to noise-prone estimation. At 10 assets and above, CVC-S consistently achieves the highest rank across all portfolio metrics. As the number of assets within the portfolio increases, the greater degree of parameterization in the modeling process of CVC-S produces various out-of-sample benefits including improved risk-adjusted returns and performance benefits robust against non-linear payoffs.

4.2.3. Portfolio re-balancing analysis

Our investigation re-calculates the desired target weights every month and the portfolio is re-balanced accordingly. In such a multi-period setting, adjustments to portfolio weights are due to the

[8] Goetzmann et al. (2007) test seven alternative measures of portfolio performance and assess their vulnerability to manipulation with a number of simple dynamic strategies. As recommended by Goetzmann et al. (2007), we use a relative risk aversion parameter of 3.

Table 5
Portfolio re-balancing analysis across out-of-sample copula-based portfolio strategies.

Metric	Method	Portfolio size, N					
		3	6	9	10	11	12
Variance	SC-N	112.62	15.07	0.57	0.23	0.16	0.03
	CVC-N	116.74	9.15	0.84	0.15	0.14	0.03
	SC-S	108.25	8.33	0.66	0.18	0.10	0.04
	CVC-S	107.50	16.58	0.71	0.17	0.12	0.04
	MVN	113.12	10.55	1.49	0.23	0.23	0.07
Maximum positive adjustment	SC-N	7.92	8.59	5.18	5.11	6.26	3.06
	CVC-N	8.51	6.76	5.32	4.09	3.64	3.24
	SC-S	7.93	8.14	5.30	4.74	4.00	3.39
	CVC-S	7.09	6.56	5.29	4.28	4.23	3.48
	MVN	7.92	6.35	6.04	3.28	4.05	3.03
Maximum negative adjustment	SC-N	−8.57	−7.17	−2.70	−2.18	−1.86	−0.86
	CVC-N	−9.24	−6.45	−2.74	−2.20	−1.70	−1.46
	SC-S	−8.49	−4.68	−2.59	−1.82	−1.89	−1.33
	CVC-S	−7.60	−6.40	−2.50	−1.87	−1.77	−1.50
	MVN	−8.57	−6.44	−5.06	−3.19	−3.12	−2.72

This table shows the variance, maximum positive, and maximum negative asset re-balancing adjustments for each portfolio strategy out-of-sample. SC-N is the Clayton standard copula (SC) with normal marginals, CVC-N is the Clayton canonical vine copula (CVC) with normal marginals, SC-S is the Clayton SC with Skew-T marginals, CVC-S is the Clayton CVC with Skew-T marginals and MVN is the multivariate normal model (benchmark case).

volatility of out-of-sample asset returns and changes in investment decisions. Since we use the same out-of-sample data for each portfolio size, the adjustments to portfolio weights capture the varying changes in investment decisions made by each portfolio strategy. Large portfolio adjustments due to re-balancing is undesirable for two reasons. First, they are difficult to implement and can undermine the feasibility of a portfolio strategy. Second, other things equal, a superior strategy should require smaller adjustments to asset weights rather than large volatile changes each period.

Table 5 shows the variance, maximum positive, and maximum negative portfolio asset weight adjustments required to re-balance to the desired target weight for each strategy. For example, in portfolios of 11 assets, the CVC-S model produces a maximum positive target weight adjustment of 4.23 (i.e., the weight of one portfolio asset moves from −1.50 to 2.73) and a maximum negative target weight adjustment of −1.77 (i.e., the weight of one portfolio asset moves from 1.23 to −0.54). For portfolios of 10 assets and above, CVC models generally exhibit lower variance in weight adjustments compared to SC models. At nine assets and above, the MVN benchmark has the highest variance in portfolio weight adjustments. This higher variance is undesirable in practice and might lead to higher turnover. Also, the magnitude of MVN's negative portfolio weight adjustment is much larger relative to the other strategies. This suggests that MVN might require more downward adjustments than other approaches. Finally, we observe that the degree of variance and the maximum positive or negative adjustment in portfolio weights required to execute each strategy tends to decrease as the size of the portfolio increases.

4.2.4. Economic performance

Table 6 reports three alternative economic metrics across the portfolio management strategies. Specifically, we model terminal wealth by hypothetically investing \$100 at the start of the out-of-sample periods for each portfolio management strategy. To gauge the feasibility of implementing each strategy, we also calculate the average turnover requirement and the effect of transaction costs on each portfolio. The average turnover is calculated as the average sum of the absolute value of the trades across the N available assets following DeMiguel et al. (2009):

$$\text{Average turnover} = \frac{1}{T-M}\sum_{t=1}^{T-M}\sum_{j=1}^{N}(|w_{k,j,t+1} - w_{k,j,t^+}|) \tag{23}$$

where N is the total number of assets in the portfolio, T is total length of the time series, M is the sample period used to parameterize the forecast models, $w_{k,j,t+1}$ is the desired target portfolio weight for asset j at time $t+1$ using strategy k and w_{k,j,t^+} is the counterpart portfolio weight before re-balancing. Similar to DeMiguel et al. (2009), we apply proportional transaction costs of 1 basis point per transaction (as assumed in Balduzzi and Lynch (1999) based on studies of transaction costs by Fleming et al. (1995) for trades on futures contracts on the S&P 500 index).

For portfolios of 10 assets and above, CVC-S produces the largest terminal wealth-regardless of whether transaction costs are included or not. Even though CVC-S's re-balancing decisions require higher turnover compared to the other strategies, the higher costs are still outweighed by the greater economic benefits. SC-S is the second best performing strategy irrespective of transaction costs and exhibits much lower turnover requirements compared to CVC-S. While SC-N performs well for small portfolios of three assets, at six assets and above, it exhibits the lowest final portfolio values and has the lowest turnover requirements in portfolios of nine assets and above. For portfolios of 6 and above, CVC-N and CVC-S outperform SC-N and SC-S, respectively. This shows that the higher degree of parameterization of CVC models leads to performance benefits above the simpler SC model for larger portfolios.

4.2.5. Value-at-Risk (VaR) backtests

Table 7 shows the performance of our portfolio management strategies using a range of VaR backtests at the 1% level, similar to Basel (2004) requirements. During each out-of-sample period, a VaR violation is recorded when the portfolio strategy return is less than the 1% VaR value of the total forecast return series for all constituent assets within the portfolio (Christoffersen, 2012). The Percentage of Failure Likelihood Ratio (PoFLR) and Conditional Coverage Likelihood Ratio (CCLR) are test statistics designed by Kupiec (1995) and Christoffersen (2012), respectively. The PoFLR focuses on the property of unconditional coverage whereas the CCLR incorporates both unconditional coverage and independence testing. Intuitively, tests of unconditional coverage indicate the magnitude of the difference between the actual and promised percentage of VaR violations in a risk management model and independence testing indicates the existence of serial VaR violations in a row. Large PoFLR and CCLR values are indicative that the proposed risk or portfolio management strategy systematically understates or overstates the portfolio's underlying level of risk.

Table 6
Economic measures of out-of-sample performance of copula-based portfolio strategies.

Economic metric	Method	Portfolio size, *N*					
		3	6	9	10	11	12
Terminal wealth exc. transaction cost	SC-N	199.18	24.03	90.08	67.52	49.51	167.96
	CVC-N	86.36	101.54	336.93	210.54	151.43	338.45
	SC-S	72.60	34.86	54.14	159.50	65.99	385.11
	CVC-S	53.41	109.90	190.00	276.84	283.71	524.93
	MVN	187.41	117.00	266.36	177.73	150.56	376.20
Average turnover	SC-N	2.15	2.32	1.75	1.30	1.43	1.21
	CVC-N	2.12	2.29	1.83	1.46	1.65	1.50
	SC-S	2.47	2.43	2.20	1.55	1.61	1.32
	CVC-S	2.43	2.65	2.26	1.86	1.75	1.65
	MVN	1.62	1.87	1.87	1.58	1.77	1.61
Terminal wealth inc. transaction cost	SC-N	180.77	21.65	83.26	63.67	46.43	159.06
	CVC-N	78.50	91.60	310.26	197.12	140.62	316.36
	SC-S	64.97	31.25	49.04	148.77	61.37	362.88
	CVC-S	47.88	97.55	171.61	254.64	262.17	487.34
	MVN	174.24	107.53	244.87	165.55	139.01	349.98

This table shows the hypothetical terminal wealth generated by each portfolio management strategy. Terminal wealth is modeled as the final portfolio value (either excluding or including transaction costs) assuming an initial investment $100 at the start of the out-of-sample period for each strategy. The turnover required to implement each strategy is also reported and can be interpreted as the average percentage of portfolio wealth traded in each period. The final portfolio value including transaction costs assumes transaction costs of 1 bps per transaction. SC-N is the Clayton standard copula (SC) with normal marginals, CVC-N is the Clayton canonical vine copula (CVC) with normal marginals, SC-S is the Clayton SC with Skew-T marginals, CVC-S is the Clayton CVC with Skew-T marginals and MVN is the multivariate normal model (benchmark case).

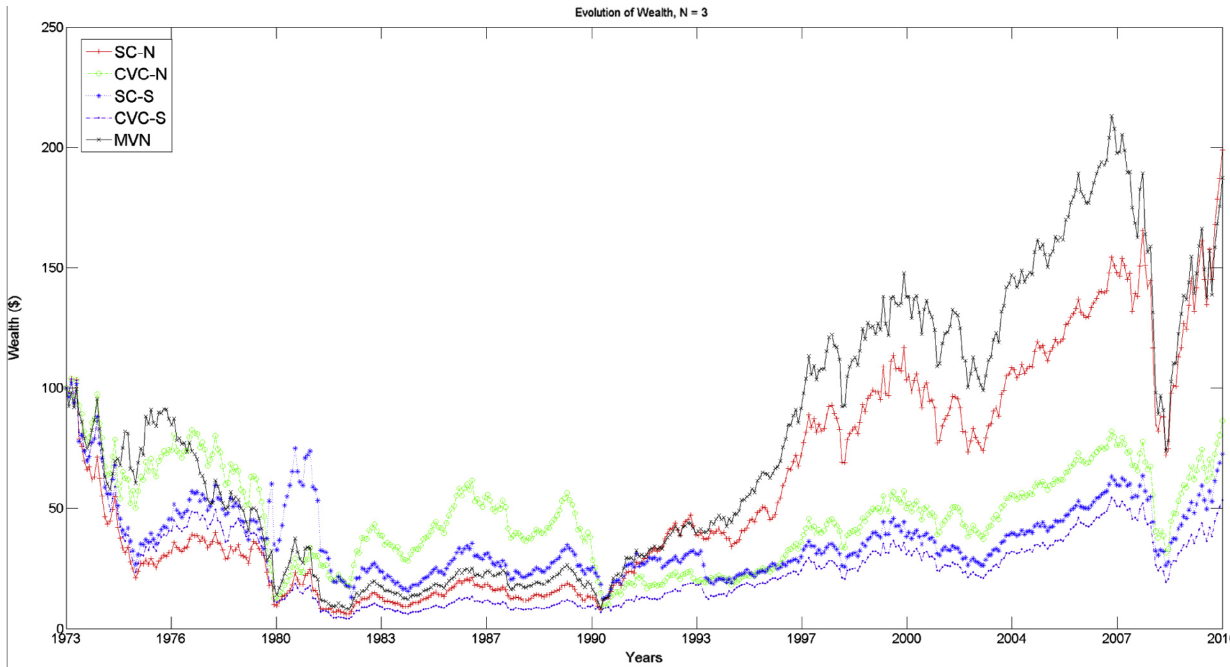
Table 7
Value-at-Risk (VaR) backtests across copula-based portfolio strategies.

VaR backtest metric	Method	Portfolio size, *N*					
		3	6	9	10	11	12
Percentage of failure likelihood ratio	SC-N	42.22	45.70	15.37	15.37	12.98	5.04
		(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.02)
	CVC-N	42.22	32.32	12.98	8.67	5.04	3.52
		(0.00)	(0.00)	(0.00)	(0.00)	(0.02)	(0.06)
	SC-S	42.22	32.32	10.75	10.75	5.04	1.20
		(0.00)	(0.00)	(0.00)	(0.00)	(0.02)	(0.27)
	CVC-S	38.82	20.54	5.04	3.52	2.23	0.46
		(0.00)	(0.00)	(0.02)	(0.06)	(0.14)	(0.50)
	MVN	29.21	15.37	2.23	2.23	3.52	2.23
		(0.00)	(0.00)	(0.14)	(0.14)	(0.06)	(0.14)
Conditional coverage likelihood ratio	SC-N	44.15	64.67	15.37	22.04	12.98	5.04
		(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.08)
	CVC-N	46.58	51.91	13.56	13.14	5.04	10.32
		(0.00)	(0.00)	(0.00)	(0.00)	(0.08)	(0.01)
	SC-S	46.58	35.41	10.75	10.75	5.04	1.20
		(0.00)	(0.00)	(0.00)	(0.00)	(0.08)	(0.55)
	CVC-S	39.36	22.60	5.04	3.52	2.23	0.46
		(0.00)	(0.00)	(0.08)	(0.17)	(0.33)	(0.80)
	MVN	32.77	22.04	10.05	10.05	10.32	10.05
		(0.00)	(0.00)	(0.01)	(0.01)	(0.01)	(0.01)
Traffic light classification	SC-N	Red	Red	Red	Red	Red	Yellow
	CVC-N	Red	Red	Red	Yellow	Yellow	Yellow
	SC-S	Red	Red	Red	Red	Yellow	Yellow
	CVC-S	Red	Red	Yellow	Yellow	Yellow	Green
	MVN	Red	Red	Yellow	Yellow	Yellow	Yellow

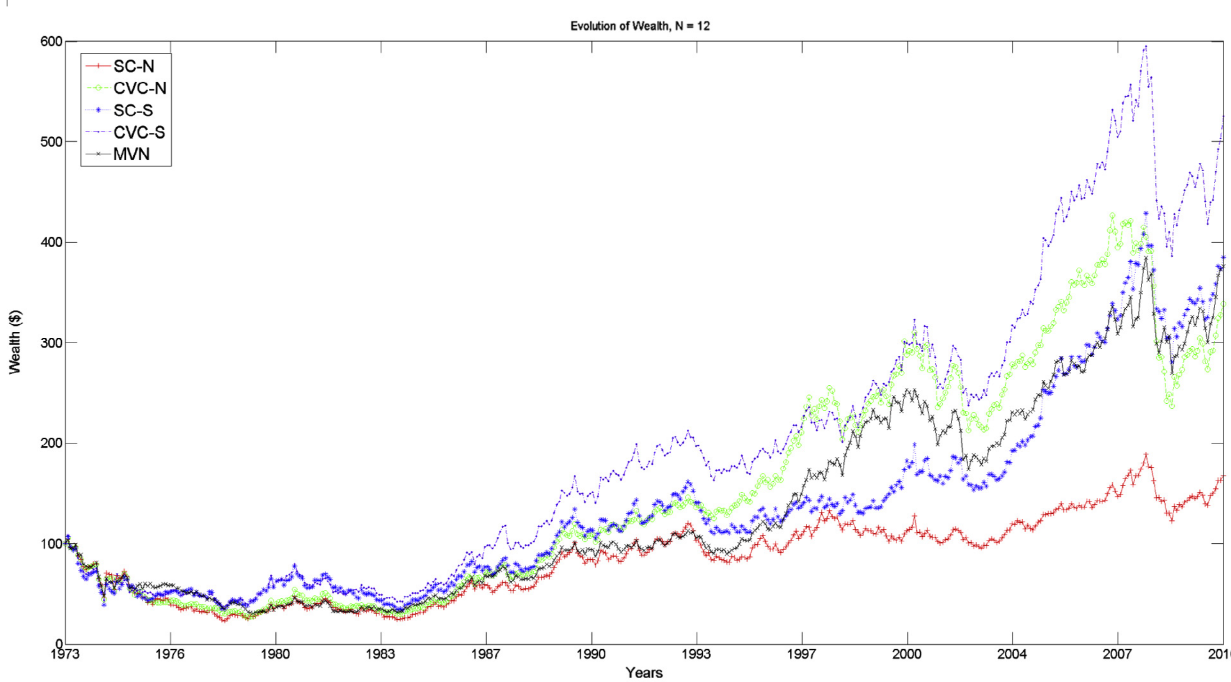
This table reports VaR backtests performed at the 1% level. The Percentage of Failure Likelihood Ratio (Kupiec, 1995) measures only unconditional coverage. The conditional coverage test (Christoffersen, 2012) is a simultaneous test of both the unconditional coverage and independence properties of VaR violations. The Traffic light approach is taken from the Basel II regulatory framework where models are categorized as 'Red': unacceptable, 'Yellow': uncertain and 'Green': acceptable. SC-N is the Clayton standard copula (SC) with normal marginals, CVC-N is the Clayton canonical vine copula (CVC) with normal marginals, SC-S is the Clayton SC with Skew-T marginals, CVC-S is the Clayton CVC with Skew-T marginals and MVN is the multivariate normal model (benchmark case).

Therefore, a superior strategy results in a test statistic closest to zero. Following Christoffersen (2012), we report the p-values for these test statistics where the null hypothesis is that the portfolio/risk management model is correct on average.

Generally, we find that CVC-S and MVN are the best performing models across all portfolio sizes and that there is a large improvement when there are nine assets or more. At 12 assets, CVC-S shows a substantial performance improvement for PoFLR compared to MVN. For portfolios less than 12 assets, the PoFLR test statistic indicates similar performance between CVC-S and MVN. However, when we account for the independence property of VaR backtests using CCLR, CVC-S exhibits superior performance compared to MVN. We also observe that when the independence property is accounted for in the testing, the performance of MVN



(a) N=3



(b) N=12

Fig. 4. Pattern of wealth accumulation for out-of-sample copula-based portfolio strategies. This figure shows the accumulation of wealth from an initial hypothetical investment of $100 in each portfolio strategy at the start of the out-of-sample period for 3-asset and 12-asset portfolios. SC-N is the Clayton standard copula (SC) with normal marginals, CVC-N is the Clayton canonical vine copula (CVC) with normal marginals, SC-S is the Clayton SC with Skew-T marginals, CVC-S is the Clayton CVC with Skew-T marginals and MVN is the multivariate normal model (benchmark case).

deteriorates across all portfolio sizes. For each of the other portfolio strategies that incorporate returns asymmetry in some form, we do not always observe an increase in the CCLR test statistic. In fact, for CVC-S, incorporation of independence testing has only a negligible impact, particularly in portfolios above six assets. Therefore, incorporation of return asymmetry in forecasting improves the independence property as the likelihood of having a sequence of VaR violations is reduced.

Based on a 10%[9] significance level for the PoFLR test statistic, CVC-S (MVN) is acceptable for portfolios of 11 and 12 assets (9, 10 and 12 assets). Based on the same statistic, SC-S is also acceptable for portfolios of 12 assets. However, applying the same significance

[9] Christoffersen (2012) recommends the use of a 10% significance level for practical risk management purposes because Type II errors (i.e., a failure to reject an incorrect model) can be very costly.

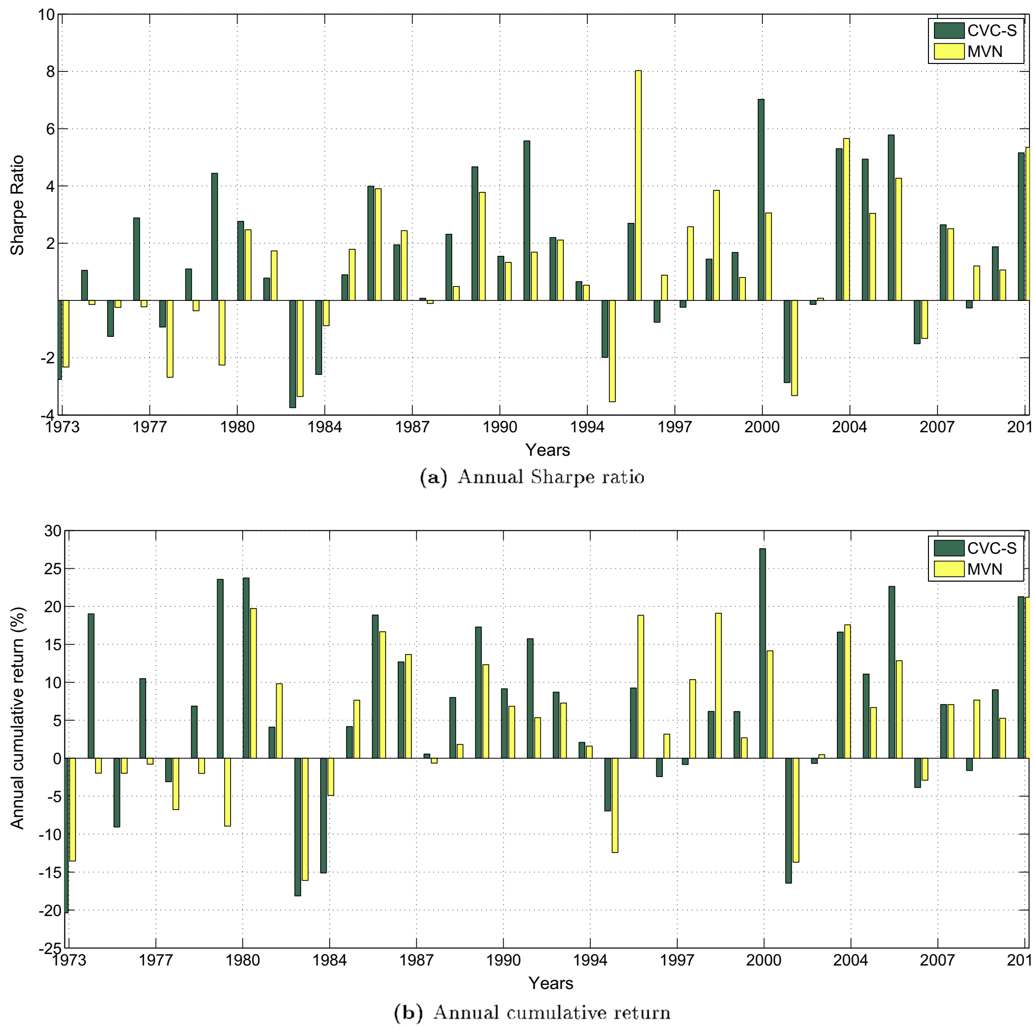


Fig. 5. Annual Sharpe ratio and cumulative return, out-of-sample, for CVC-S vs. MVN models This figure shows annual comparisons between the Sharpe ratio and annual cumulative returns between the CVC-S and MVN models. CVC-S is the Clayton canonical vine copula (CVC) with Skew-T marginals and MVN is the multivariate normal model (benchmark case).

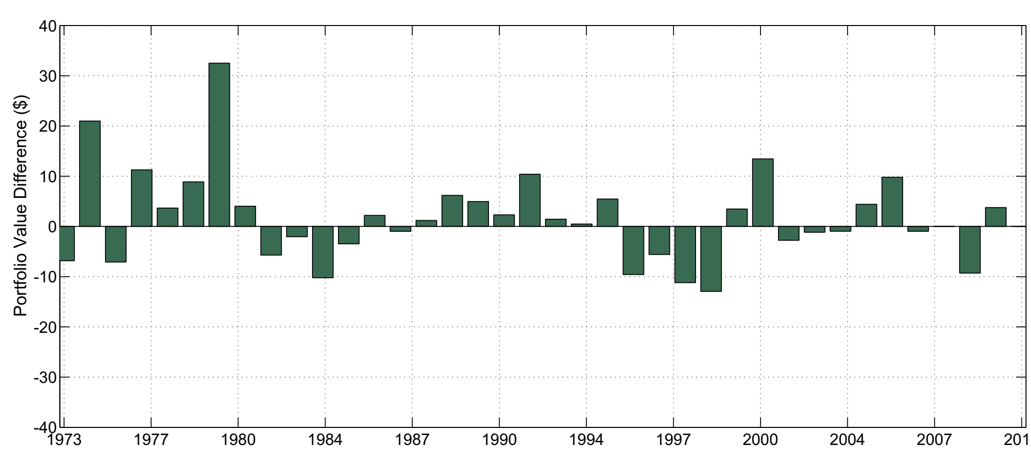


Fig. 6. Difference in end of year portfolio value, out-of-sample, for CVC-S vs. MVN models. This figure shows the difference in end of year portfolio values between CVC-S and MVN annually. The end of year portfolio value of MVN is subtracted from CVC-S, based on a hypothetical investment of $100 in each strategy at the beginning of each year. CVC-S is the Clayton canonical vine copula (CVC) with Skew-T marginals and MVN is the multivariate normal model (benchmark case).

Table 8
Average annual out-of-sample performance differential between CVC-S vs. MVN models.

Metric	Whole	Normal	Crisis
Sharpe ratio	0.33	0.27	0.56
Sortino ratio	0.63	0.49	1.14
Portfolio value	1.59	0.91	4.11

This table shows the average annual differential between CVC-S and MVN (i.e., CVC-S minus MVN) for the Sharpe ratio, Sortino ratio, and end-of-year portfolio values generated annually by hypothetically investing $100 at the start of each year. 'Whole' denotes all years in the out-of-sample dataset from 1973 to 2010. 'Crisis' denotes the annual subperiods constituting the bottom quintile of US stock market returns (i.e., the 8 worst performing years out of the entire 37 year out-of-sample period) and 'Normal' denotes the remaining annual subperiods in the out-of-sample dataset. CVC-S is the Clayton canonical vine copula (CVC) with Skew-T marginals and MVN is the multivariate normal model (benchmark case).

level to the CCLR statistic (notably a stricter test), CVC-S is an acceptable model for portfolios of 10 assets and above, while MVN is now rejected across all portfolio sizes. SC-S remains acceptable using the CCLR test statistic for a portfolio of 12 assets.

As a third form of analysis we apply the 'traffic light' approach, taken from Basel (2004) in which risk management models are classified into three categories depending on the number of VaR violations. For our scenario of 450 out-of-sample periods, models with 13 or more VaR violations are within the 'Red' category. Models within the 'Green' category have less than 6 violations, those within the 'Yellow' category have between 6 to 12 violations. We find that all strategies perform poorly for portfolios of three and six assets. However, CVC-S and MVN improve dramatically for portfolio sizes of 9 and above. CVC-S is the only strategy that achieves a 'Green' zone classification-for the 12-asset portfolio.

Generally, for portfolios of three and six assets, the multivariate probability models do not perform well when VaR backtests are considered. However, our results continue to support the view that for portfolios of 10 assets and above, the CVC-S strategy improves portfolio decisions as there is reduced frequency and increased independence of VaR violations. This conclusion comes from the CCLR and the Basel (2004) traffic light tests.

4.2.6. Further analysis of time-series performance

Fig. 4 shows the accumulation of wealth for all the strategies when the portfolio contains either three assets or 12 assets. In Fig. 4a, analyzing portfolios of three assets, from 1973 to 1990, the portfolio management strategies perform similarly. Beyond 1991, simple portfolio strategies such as SC-N and MVN start to outperform the other models. MVN tends to outperform SC-N from 1993 onwards but experiences large losses in 2007. SC-N experiences lower losses and is able to recover its portfolio value from 2008 to 2010 to outperform MVN. Based on this analysis, for small portfolios the Clayton SC captures lower tail dependence adequately and implementing the more complicated Clayton CVC is unnecessary.

We see in Fig. 4b analyzing portfolios of 12 assets, from 1973 to 1987, all portfolio strategies perform similarly. From 1987 onwards, CVC-S begins to exhibit economic superiority by producing returns above those of other models. From 1993 onwards, CVC-N also begins to exhibit relative superiority over the other models (except for CVC-S). This figure shows that controlling for lower tail dependence using the Clayton CVC and asymmetries within the marginals, the portfolio has the ability to insulate downside risk and, to some extent, protect the value of the portfolio with little loss to upside return. Within our dataset after 1987, all indices exhibit high levels of negative skewness, whereas before 1987, about half the indices exhibit positive skewness. Thus, the use of CVC-S results in improved portfolio management when negative skewness is prominent. SC-N performs poorly for large portfolios as the single asymmetric dependence parameter in the Clayton SC asymptotes towards zero due to the size of the portfolio. However, as CVC-N captures asymmetries within the marginals it is less affected by the dilution of the asymmetric dependence parameter for large portfolios.

Fig. 5 shows the annual Sharpe ratios and cumulative annual returns for each year out-of-sample, focusing on CVC-S and MVN for portfolios of 12 assets. We can see that CVC-S often produces Sharpe ratios and upside gains greater than that of MVN. Notably, during the years 2000 onwards, compared to MVN, CVC-S mainly produces larger or similar Sharpe ratios and annual cumulative returns.

Fig. 6 shows the difference in (hypothetical) end of year portfolio values between CVC-S and MVN (based on a hypothetical investment of $100 in each strategy at the beginning of each year). The difference in values at the end of each year is obtained by subtracting the portfolio value of MVN from CVC-S. Generally, we can see that CVC-S produces greater economic returns than MVN over all. Moreover, a majority of years favor the CVC-S strategy and the magnitude of the value difference tends to be higher in years when CVC-S outperforms MVN.

Table 8 shows the average annual differential across three alternative portfolio metrics for CVC-S minus MVN for the 'Whole', 'Crisis' and 'Normal' periods within the out-of-sample study. The 'Whole' period denotes the entire out-of-sample period, 1973 to 2010. The 'Crisis' years are identified as the bottom quintile of US market index monthly returns-that is, the 8 years that exhibit the largest frequency of the worst performing months.[10] The 'Normal' period consists of the remaining 29 years.

Across the entire out-of-sample period, on an average annual basis, CVC-S outperforms MVN by 0.33 when applying the Sharpe Ratio. Given a focus on downside risk, the Sortino Ratio indicates that CVC-S delivers a substantial performance advantage over MVN: the differential is 0.63. To complete the overall comparison, on average, CVC-S results in a higher dollar value of $1.59 per year based on hypothetically investing $100 at the start of each year.

Asymmetric dependence or excessive downside correlation across equity returns is more prevalent during bear markets or 'Crisis' periods. Thus intuitively, during such periods a strategy that explicitly manages asymmetric dependence should exhibit superior performance compared to strategies that do not. Interestingly, while we find that CVC-S exhibits superior performance during both 'Crisis' and 'Normal' periods the superiority is greater during the 'Crisis' period. Specifically, during this part of our sample the Sharpe Ratio of CVC-S is larger than MVN by a magnitude of 0.56, compared to 0.27 during the 'Normal' period. This effect is even more pronounced when we focus on downside risk: the difference in the Sortino Ratio is 1.14 in favor of the copula-based strategy during 'Crisis' years and this differential is more than twice the value observed during 'Normal' years. Finally, when we focus on the average end-of-year portfolio value differences, CVC-S has a higher value of $4.11 during the 'Crisis' period compared to a smaller superiority of $0.91 during 'Normal' years (relative to $100 hypothetical investments occurring at the beginning of each year).

These results show further evidence that CVC-S, the strategy that incorporates asymmetric dependence using the Clayton CVC and skewness within the marginals, is able to produce superior forecasts of equity returns compared to competing models, leading

[10] The 8 'Crisis' years identified are 1974, 1975, 1979, 1981, 1982, 1987, 1990 and 2008.

to improved portfolio allocation decisions and enhanced performance.

5. Conclusion

In this study, we investigate whether using asymmetric copula models to forecast returns for portfolios ranging from 3 to 12 assets can produce superior investment performance compared to traditional models. We examine the efficient frontiers produced by each model and focus on comparing two methods for incorporating scalable asymmetric dependence structures across asset returns using the Archimedean Clayton copula in an out-of-sample, long-run multi-period investment. As traditional Mean–Variance Portfolio Theory (MVPT) does not account for asymmetry in returns distributions, it is quite plausible that there is a need for more advanced portfolio management strategies that incorporate asymmetries within the forecasting process and during the optimization of the investor's utility function.

We find evidence that for portfolios of 10 assets and above, the Clayton CVC outperforms the Clayton standard copula (SC) across a broad range of metrics over a long-run, multi-period horizon. The most advanced model we implement, in which asymmetries within the dependence structure and marginals are modeled using the Clayton CVC and skewed Student *t* of Hansen (1994) (CVC-S), consistently produces statistical and economically significant gains superior to the other models tested, including the multivariate normal (MVN) model. Despite the strategy having high turnover requirements, even when transaction costs are incorporated, there are greater economic benefits relative to the other strategies. CVC-S also exhibits the best performance when a series of Value-at-Risk (VaR) backtests are applied to larger portfolios. Furthermore, it is able to consistently generate strong positively skewed returns for larger portfolios-a portfolio characteristic that is highly attractive to most rational investors. While the superiority of the CVC-S strategy over the traditional symmetric MVPT approach is generally seen across our sample, it is strongest during 'Crisis' years. This finding suggests that the CVC-S approach successfully manages asymmetric dependence compared to the other models tested.

In addition, our analysis shows that as the number of assets increases within the portfolio, modeling of the dependence structure across the assets has a greater impact. For smaller portfolios, modeling the asymmetry within the marginals themselves plays a more crucial role. The Clayton CVC produces superior statistical and economic outcomes compared to the Clayton SC for portfolios of six assets and above. Accordingly, we conclude that CVC copulas are 'worth it' when managing portfolios of high dimensions due to their ability to better capture asymmetries within the dependence structure than either the SC copula or multivariate normality models.

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